KANTONSSCHULE IM LEE
WINTERTHUR

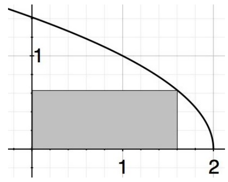
MATURITÄTSPRÜFUNGEN 2012

Klasse:	**4h**
Profil:	M/MN/N
Lehrperson:	Rolf Kleiner

MATHEMATIK
in englischer Sprache (immersiv)

Zeit:	3 Stunden *3 hours*
Erlaubte Hilfsmittel:	Grafiktaschenrechner ohne CAS, beliebige Formelsammlung *Graphing calculator without CAS, formula booklet of your choice*
Bemerkungen:	Die Prüfung enthält 6 Aufgaben mit 50 Punkten. *The exam consists of 6 problems with 50 marks.* Lösen Sie jede Aufgabe auf ein separates A4-Blatt. *Solve each problem on a separate piece of paper.* Schreiben Sie Ihre Lösungswege klar nachvollziehbar auf. *Show all your working.* Geben Sie numerische Ergebnisse wenn möglich exakt, andernfalls sinnvoll gerundet an. *Give exact values for your numerical answers, if possible. Otherwise, round your results appropriately.*

1 [4m] A rectangle is inscribed in the region enclosed by the graph of $y=\sqrt{2-x}$ and the coordinate axes (see figure). Find the exact value of the largest area possible of such a rectangle by using a derivative.



2 [8m] $f(x)=a\cdot x^3+b\cdot x^2+c\cdot x+d$ is a polynomial function of degree 3. The graph of *f* has a local minimum or maximum at $(0,0)$ and a point of inflection at $(1,2)$.

a) Determine the coefficients *a*, *b*, *c* and *d* and hence an equation for the function *f*.

In case you have not been able to solve problem **a)** you may use $f_2(x)=-x^3+3x^2$ instead of *f* to solve problems **b)** and **c)**.

b) Determine the area enclosed between the graph of *f* and the *x*-axis.

c) The straight line *l* through the minimum and maximum point of the graph of *f* intersects the graph of *f* in a further point *S*. Find the coordinates of *S* as well as the angle of intersection between the line *l* and the graph of *f* at the point *S*.

3 [7m] The curves of $y=\sin(x)\cdot\cos(x)$ and $y=\log_{10}(x+1)$ intersect at the origin $(0,0)$ and at a point $P(x_0,y_0)$ in the first quadrant.

a) Newton's method – applied appropriately – may be useful in finding an approximation for x_0. Show that one step of Newton's method leads from $x_1=1.0$ to $x_2\approx1.2$ (which is, in fact, a better approximation for x_0).

b) Let *A* be the area of the region enclosed by the two curves in the first quadrant. Use the result of problem **a)** to find an approximation for the area *A* by applying the parabola method (Simpson) with 4 subintervals.

4 [13m] Given are the points $A(9,-2,7)$, $C(1,2,-9)$ and $P(12,9,3)$ as well as the straight line *l*: $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}7\\10\\0\end{pmatrix}+t\cdot\begin{pmatrix}-2\\2\\1\end{pmatrix}$.

a) Find the distance of the plane $\Pi(ACP)$ from the origin $(0,0,0)$.

b) A ray of light in the direction of the line *l* is reflected in the plane $\Pi(ACP)$. Find the angle between the directions of the incoming and the reflected ray of light without calculating the direction of the reflected ray.

c) Find the coordinates of point *B* on the line *l*, such that *ABC* is a right-angled triangle with $\gamma=\sphericalangle ACB=90°$.

d) Find the coordinates of the point *L* on the line *l*, such that *L* is equidistant from the points *A* and *C*.

e) The points *A* and *C* lie on a sphere, the centre of which lies on the line *l*. Use the result of problem **d)** to give an equation of this sphere and to show that the point $P(12,9,3)$ lies inside this sphere.

5 [13m] Ten balls are numbered from 1 to 10 and randomly placed in a row.

a) Find the probability that the five odd-numbered balls (1, 3, 5, 7, 9) will lie next to each other *if* the five even-numbered balls (2, 4, 6, 8 and 10) are lying next to each other.

b) Show that the probability of the five even-numbered balls lying next to each other is $\frac{1}{42}$.

Use the result of problem **b)** to solve problems **c)**, **d)** and **e)**.

c) The ten balls are randomly placed in a row 50 times. Find the probability that the five even-numbered balls will lie next to each other more than twice.

d) Determine how many times the ten balls need to be randomly placed in a row so that the probability of the five even-numbered balls lying next to each other at least once is higher than 90%. Use algebraic methods to solve a suitable inequality.

e) In a game, I will win 100 Fr. if the five even-numbered balls lie next to each other. If not, I get a second chance and will win 50 Fr. if this event happens in the second row. Otherwise, I lose 2 Fr. If X denotes the number of francs won in a game, find the expected value and the standard deviation of X.

6 **a)** [3m] Show that the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$ diverges

by demonstrating with a rough figure that $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}>\int_1^n \frac{1}{x}dx$

and by then showing that the improper integral $\int_1^\infty \frac{1}{x}dx$ diverges.

b) [2m] A worm crawls along an elastic rope of initial length 100 cm from the beginning towards the end of the rope in the following manner:

Step 1A: The worm crawls 10 cm, which is $\frac{1}{10}$ of the length of the rope.

Step 1B: The rope is stretched by 100 cm. So, the worm is now 20 cm away from the beginning and 180 cm away from the end of the rope.

Step 2A: The worm crawls another 10 cm along the rope, which is $\frac{10}{200}$ or $\frac{1}{20}$ of the present length of the rope.

Step 2B: The rope is stretched by another 100 cm to a total length of 300 cm. So, the worm is now $\frac{300}{200}\cdot(20+10)=45$ cm away from the beginning and $300-45=255$ cm away from the end of the rope.

This process (worm crawling 10 cm then rope being stretched by 100 cm) is repeated over and over again. Show that the worm *will* eventually reach the end of the rope by considering the relative distances travelled in each step, i.e. the ratios

$$\frac{\text{distance travelled by the worm in a particular step}}{\text{total length of the rope at that moment}}.$$

1 $A(x) = x \cdot \sqrt{2-x}$; maximum area if $A'(x) = 0$ [1↓] : $1 \cdot \sqrt{2-x} + x \cdot \frac{(-1)}{2\sqrt{2-x}} = 0$ [2↓]

$\Rightarrow x = \frac{4}{3}$ [3↓] $\Rightarrow$ $\boxed{A = \frac{4}{9} \cdot \sqrt{6} \approx 1.09}$ [4P]

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2 a) $\left| \begin{array}{r} f(0)=0 \\ f'(0)=0 \\ f(1)=2 \\ f''(1)=0 \end{array} \right|$ [1↓] : $\left| \begin{array}{r} d=0 \\ c=0 \\ a+b+c+d=2 \\ 6a+2b=0 \end{array} \right|$ [2↓] $\Rightarrow a=-1,\ b=3 \Rightarrow$ $\boxed{f(x) = -x^3 + 3x^2}$ [3P]

b) $\int_0^3 \left(-x^3+3x^2\right) dx$ [1↓] $= \left[-\frac{1}{4}x^4 + x^3\right]_0^3 =$ $\boxed{\frac{27}{4} = 6.75}$ [2P]

c) $y' = -3x^2 + 6x = 0 \Rightarrow x_1 = 0,\ x_2 = 2$ and $(0,0),\ (2,4)$ [1↓] $\Rightarrow$ line $y = 2x$

intersection with curve at $\boxed{S(1,2)}$ [2P] ;

$f'(1) = 3 \Rightarrow \measuredangle = \arctan(3) - \arctan(2) \approx$ $\boxed{8.13°}$ [1P]

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3 a) Newton function $f(x) = \sin(x)\cdot\cos(x) - \log_{10}(x+1)$

$x_{n+1} = x_n - \dfrac{\sin(x_n)\cdot\cos(x_n) - \log_{10}(x_n+1)}{\cos^2(x_n) - \sin^2(x_n) - \dfrac{1}{\ln(10)\cdot(x_n+1)}}$ [3↓]

$\Rightarrow x_2 = 1 - \dfrac{\sin(1)\cdot\cos(1) - \log_{10}(2)}{\cos^2(1) - \sin^2(1) - \dfrac{1}{\ln(10)\cdot 2}} \approx$ $\boxed{1.2426 \approx 1.2}$ [4P] [more precise: 1.195]

b) $A \approx \frac{1.2-0}{3\cdot 4}\cdot(0 + 4\cdot(\sin 0.3\cdot\cos 0.3 - \lg 1.3) + 2\cdot(\sin 0.6\cdot\cos 0.6 - \lg 1.6) +$

$+4\cdot(\sin 0.9\cdot\cos 0.9 - \lg 1.9) + (\sin 1.2\cdot\cos 1.2 - \lg 2.2))$ [2↓] $=$

$0.1\cdot(0 + 4\cdot 0.1684 + 2\cdot 0.2619 + 4\cdot 0.2082 + (-0.0047)) \approx$ $\boxed{0.202}$ [3P]

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4 a) $\Pi(ACP):\ 8x - 4y - 5z - 45 = 0$ [2↓] $\Rightarrow \mathrm{d}(\Pi, O) = \frac{|-45|}{\sqrt{105}} =$ $\boxed{\frac{3\cdot\sqrt{105}}{7} \approx 4.39}$ [3P]

b) $\cos\left(\frac{\varphi}{2}\right) = \frac{\left|\vec{n}_\Pi \circ \vec{d}_l\right|}{\left|\vec{n}_\Pi\right|\cdot\left|\vec{d}_l\right|}$ [1↓] $= \frac{\left|\begin{pmatrix}8\\-4\\-5\end{pmatrix}\circ\begin{pmatrix}-2\\2\\1\end{pmatrix}\right|}{\sqrt{105}\cdot\sqrt{9}} = \frac{29}{\sqrt{105}\cdot 3} \Rightarrow \varphi \approx$ $\boxed{38.75°}$ [3P]

c) Point B on l: $(7-2t,\ 10+2t,\ t)$; $\overrightarrow{AC} \circ \overrightarrow{BC} = 0$: $\begin{pmatrix}-8\\4\\-16\end{pmatrix} \circ \begin{pmatrix}2t-6\\-2t-8\\-t-9\end{pmatrix} = 0$ [1↓]

[or: B where line l intersects normal plane to AC through C: $2x - y + 4z + 36 = 0$]

$\Rightarrow t = 20 \Rightarrow$ $\boxed{B(-33,\ 50,\ 20)}$ [2P]

d) Point L on l: $(7-2t,\ 10+2t,\ t)$; equidistant plane of AC: $2x-y+4z-6=0$ or

$\overline{AL}=\overline{CL}$: $\sqrt{(-2-2t)^2+(12+2t)^2+(-7+t)^2}=\sqrt{(6-2t)^2+(8+2t)^2+(9+t)^2}$ 2↓

$\Rightarrow t=-1 \Rightarrow$ $\boxed{L(9\,,8\,,-1)}$ 3P

e) Sphere: centre L and $r=\overline{AL}=\sqrt{164}$: $\boxed{(x-9)^2+(y-8)^2+(z+1)^2=164}$ 1P

$\overline{PL}=\sqrt{26}\ <\ \overline{AL}=r=\sqrt{164}\ \Rightarrow$ $\boxed{P \text{ lies inside the sphere}}$ 1P

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5 **a)** There are 6 ways of placing the five even balls (1 to 6, 2 to 7, …, 6 to 10).

The odd balls have to be at the beginning or at the end: $P=\frac{2}{6}=\boxed{\frac{1}{3}}$ 1P

b) $\frac{6\cdot 5!\cdot 5!}{10!}$ 2↓ or $6\cdot \text{P(first 5 balls even)}=6\cdot\left(\frac{1}{2}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7}\cdot\frac{1}{6}\right)$ 2↓ $=\boxed{\frac{1}{42}}$ 2P

c) $P(k>2)=1-P(k=0)-P(k=1)-P(k=2)$ 1↓ ; binomial distrib. with $n=50$, $p=\frac{1}{42}$;

$=1-0.2997-0.3655-0.2184=\boxed{0.116}$ 3P

d) $1-\left(\frac{41}{42}\right)^n>0.9$ 1↓ $\Rightarrow n>\frac{\lg(0.1)}{\lg\left(\frac{41}{42}\right)}\approx 95.55\Rightarrow$ $\boxed{96 \text{ rows}}$ 2P

e)

X	100	50	-2
P	$\frac{1}{42}$	$\frac{41}{42}\cdot\frac{1}{42}$	$\frac{41}{42}\cdot\frac{41}{42}$

2↓

$E(X)=100\cdot\frac{1}{42}+50\cdot\frac{41}{42}\cdot\frac{1}{42}-2\cdot\frac{41}{42}\cdot\frac{41}{42}=$ $\boxed{1.64 \text{ Fr.}}$ 3↓

$\text{Var}(X)=E\left(X^2\right)-\left(E(X)\right)^2=100^2\cdot\frac{1}{42}+50^2\cdot\frac{41}{42}\cdot\frac{1}{42}+(-2)^2\cdot\frac{41}{42}\cdot\frac{41}{42}\ -1.637^2$ +1

$\sigma(X)=\sqrt{\text{Var}(X)}=\sqrt{297.33}=$ $\boxed{17.24 \text{ Fr.}}$ 5P

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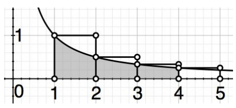
6 **a)** Example $n=4$: 

Sum of 4 rectangles $>$ Shaded area 2P ;

$\int_1^\infty \frac{1}{x}dx=\lim_{x\to\infty}(\ln(x))=\infty$ 1P $\Rightarrow 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots>\int_1^\infty\frac{1}{x}dx=\infty$ 3P

b) $\frac{1}{10}+\frac{1}{20}+\frac{1}{30}+\frac{1}{40}+\ldots=\frac{1}{10}\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\right)=\frac{1}{10}\cdot(\infty)=\infty\ >1$ (see a) 2P

[End of rope: $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\approx 10$, i.e. if $9<\ln(n)<10 \Rightarrow 8103<n<22026$]

Grading Scale and Results – 4h N,MN,M – Summer 2012											
Grade	**6**	**5.5**	**5**	**4.5**	**4**	**3.5**	**3**	**2.5**	**2**	**1.5**	**1.0**
Number of marks	40	36	32	28	24	20	16	12	8	4	0
Number of students	4	4	2	0	2	0	1	1	0	1	0